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formal values, is completely determined by the possible substitution groups on these variables. The above remarks apply directly to all rational functions of the n variables as well as to the more special functions which are commonly (but not universally) called polynomials.

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ON THE COMBINATION OF INVOLUTIONS.*

By D. N. LEHMER, University of California.

1. Given two involutions of rays in the same plane with centers at A and B a quadratic reciprocal transformation may be set up in the plane as follows: To any point P make correspond the intersection P' of the two rays at A and B which correspond in the involutions to the rays PA and PB. The point-to-point transformation thus defined is clearly involutorial. Certain points appear as exceptional in that the correspondence is not unique, namely: all of the points on the line joining A and B go by the transformation into the same point C. Further, all of the points on AC go into the same point B, and all of the points on BC go into the point A, as is seen by making the construction according to the definition. It will be suspected that the point C is the center of an involution which might be used instead of A or B to define the same transformation. That this is the case appears as a corollary from the following fundamental theorem.

2. Theorem. *If the point P describes a straight line the corresponding point P' describes a conic through A, B, and C.*

The point row P projects to A and B in two perspective pencils. The corresponding rays in the involutions at A and B are therefore projective and generate a conic through A and B. Since the point row P meets the line AB in one point the corresponding point P' goes through C. From the construction it appears further that the point row of the first order P is projective to the point row of the second order P'. It follows therefore that the pencils PC and $P'C$ are projective and that they are in involution. The point C is thus in all respects coördinate with A and B.

If we assume the theorem that a conic meets a curve of degree m in $2m$ points we may prove easily the more general theorem.

3. Theorem. *A curve of degree n goes by the above transformation into a curve of degree $2n$.*

For let the curve C of degree n go into a curve C'. Cut across C' by a line g. Transform C' and g. C' goes back into C and g goes into a conic γ. The points common to C and the conic γ, $2n$ in number, correspond to the points common to C' and the arbitrary line g.

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* Presented at the September meeting of the San Francisco Section of the American Mathematical Society, under slightly different title.

4. In the above theorems we have assumed tacitly that the curve described by P does not pass through A, B, or C. If for instance in the first theorem the line traversed by P passes through A, the point P' must traverse the line corresponding to it in the involution at A. The conic in fact degenerates into this line and the line BC which corresponds to the point A. The more general theorem of paragraph 3 should be stated as follows to take account of this degeneracy. A curve of degree n which passes k times through the points A, B, or C goes into a degenerate curve of degree $2n$, which contains the sides of the triangle counted k times and a curve of degree $2n-k$. Thus a conic goes generally into a quartic which has A, B, and C for double points since the conic cuts the sides of the triangle generally in two points. If the conic goes through the point A the corresponding curve is a cubic with double point at A. If the conic contains two points A and B the corresponding curve is a conic through A, B, and C. If the conic passes through all three points the corresponding curve is a straight line.

5. The theory of the quadratic transformation is of course perfectly well known and many ways have been devised for realizing it geometrically.[*] The above method appears in many ways the simplest and most effective. Thus the four invariant points appear as the intersections of the focal rays at A with the focal rays at B; and since the focal rays at C must go through them they are presented as the vertices of a quadrilateral of which ABC is the diagonal triangle. Again, since AB and AC are corresponding rays in the involution at A that involution is determined by one pair of corresponding points P, P'. Two transformations S and T having the same base triangle ABC produce a collineation in the plane. For S throws a line into a conic through ABC, and T throws this conic into a line again. Conversely, it is easy to show that any collineation is the product in this way of two quadratic transformations. The vertices A, B, C, are the self-corresponding points in the collineation.

6. We have seen that two involutions A and B determine a third C. We will consider now three involutions A, B, and C which are not so related. It is clear that a point P in the plane will not generally have a point P' conjugate to it with respect to all three involutions. For the rays at A, B, and C which correspond to PA, PB, and PC will not generally meet in a point. Points may be found, however, which do have conjugates with respect to all three as the following theorem shows.

7. Theorem. *On any line in the plane there is at least one point and at most three which have conjugate points with respect to three unrelated involutions.*

For let the point P traverse the line g. The rays at A and B which correspond in the involutions at those points to PA and PB generate as we have seen a conic through A and B. There is also a conic through A and C. These two conics have a point A in common, and so have at least one and at most three other common points. These are easily seen to be points of the kind in question. Thus we have the following

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[*] See Reye, *Geometry of Position*, Holgate's Translation, p. 111.

Theorem. *The locus of points which have conjugate points with respect to three unrelated involutions in the plane is a cubic curve.*

8. If a point P is on the curve the point P' is also, as appears from the character of the correspondence. The points of the curve are thus paired in involution in the sense that four points P that project to A, B, or C in four harmonic rays correspond to four points P' that also project to A, B, or C in four harmonic rays. The points A, B, and C are easily seen to be points on the curve. To show this, construct the point of intersection C' of the rays at A and B which correspond to CA and CB. C and C' are conjugate points.

9. *If PP' and QQ' are two pairs of conjugate points so also are the points $(PQ, P'Q')=R$ and $(PQ', P'Q)=R'$.* For joining PP' and QQ' to A we get two pairs of corresponding rays in the involution at A. But if we project to A the opposite vertices of the complete quadrilateral PP', QQ', RR', we get three pairs of rays in the same involution. Thus the vertices RR' project to A and similarly to B and C in the involutions at those points.

10. *If P and P' are fixed and Q is taken arbitrarily on the curve, then QP and QP' meet the curve again in conjugate points R and R'.*

If Q moves so as to make R approach P then the conjugate R' must approach P' and we have at once the beautiful

Theorem. *The tangents to the curve at a pair of conjugate points meet again on the curve.*

In this case Q' lies on the line joining PP'.

11. Consider now four unrelated involutions of rays in the same plane. Is it possible to find points in the plane conjugate with respect to all four? Let the centers be A, B, C, and D. Points conjugate with respect to A, B, and C lie on a cubic through those points. Points conjugate with respect to A, B, and D lie on a cubic through those points. The two cubics have A and B for common points and also the point M, where M is the center of the third involution equivalent to, and determined by, the involutions at A and B, as in the first paragraph. Other intersections must occur in pairs, for if P lies on both cubics so also must P', which is conjugate to it with respect to all four involutions. If they have two pairs of points in common they must have a third pair. If they have still another pair it is possible to find as many pairs as one desires by the above quadrilateral construction. In that case the curves coincide and D lies on the cubic determined by A, B, and C. It follows that *the cubic may be generated by any three points on it as centers of involution,* provided, however, the three points are not centers of related involutions.

It follows from this that *any point on the cubic is joined to three pairs of conjugate points by three pairs of rays in involution.*

12. This last theorem is indeed the fundamental starting point for the

usual treatment of the general cubic by synthetic methods.*

13. We have omitted to speak of the corresponding discussion that might be made of point rows in involution. The extension of the method to space of three dimensions is of interest, however. Consider three quadric surfaces, A, B, and C. Then to any point in space P, we make correspond the point P' of intersection of the polar planes of P with respect to the three surfaces.† The correspondence is reciprocal and generally unique. By it *a straight line goes into a twisted cubic.* The polar planes of P describe projective axial pencils. If we cut across by an arbitrary plane we get three projective pencils of rays in that plane. By a discussion precisely similar to the discussion in paragragh 7 it is shown that one set of corresponding rays at least and three sets at most will meet in a point. These are points on the locus of P'.

14. By the above transformation *a plane goes into a cubic surface.* For suppose the plane μ goes into the surface M. Cut across M by an arbitrary line g. Reciprocate and the surface M goes into the plane μ and g goes into a twisted cubic which meets μ in one or three points. These are conjugate to the three points in which g meets M.

15. By an entirely similar argument, assuming the theorem: *A twisted cubic meets a surface of degree n in 3n points,* we may show that, *A surface of degree n goes into a surface of degree* 3*n.* Also by assuming that, *A cubic surface intersects a space curve of degree n in 3n points,* we may show that, *The above transformation throws a curve of degree n into a curve of degree* 3*n.*

16. We have seen in the corresponding plane theory that a straight line is transformed into a conic, and a conic into a quartic. The particular conic that corresponds to a straight line however transforms back not into a quartic but into the line. This was explained by the fact that the conic obtained from a straight line passed through each of the vertices of the triangle ABC and thus degenerated in the transformation into a quartic made up of four lines. A similar difficulty appears in the space theory. A plane γ goes into a cubic C, and this cubic C instead of going into a surface of degree nine, as the theory requires, goes back into the plane γ again. Also the cubic curve that corresponds to a straight line goes back into a straight line again and not into a curve of degree nine. To explain this difficulty, consider the plane γ and the corresponding cubic surface C_3. Cut across C_3 by any plane α. Reciprocate C_3 and α. C_3 goes into a plane γ, and α into a cubic surface A. Therefore the curve of intersection of α with C_3 goes into a cubic curve C lying in γ. Cut C_3 by a second plane β, and let the intersection line of α and β meet C_3 in three points P, Q, and R. As before, the curve of intersection of β and C_3 goes into a cubic curve b in γ. The two curves b and c have the three points P', Q', and R' in common. They have then generally six others, S_1, S_2, S_3, S_4, S_5, S_6. It is seen that S_1 corresponds to a point on

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* See Schraeter, *Ebene Kurven Dritter Ordnung.* Leipzig, 1888.

† This correspondence is due to Steiner, *Gess. Werke, II,* p. 651.

α since it is a point on a, and also to a point on β since it is a point on b. But this means that the polar planes of S_1 meet in two distinct points and therefore all three polar planes meet not in a single point but in a line. There are six such points generally in any plane. *The locus of points therefore, such that their polar planes with respect to three quadric surfaces meet in a line, is a twisted sextic curve.* It is of course the Jacobian of the three surfaces.*

17. We note in passing that the six lines corresponding to the six points S in γ lie entirely in the surface C_3. Also that just as in the plane theory if a line goes through one of these points the corresponding twisted cubic degenerates into a conic and a straight line. Also if a line goes through two of these points the corresponding twisted cubic is made up of three straight lines. To the fifteen lines therefore joining the six points S in pairs, correspond fifteen other lines lying in the surface. Further, the conics through any five correspond to degenerate curves of degree six made up of six straight lines, five of which are the lines corresponding to the points S. The sixth also, therefore, lies on the surface giving five more or twenty-seven in all. The usual theorems may of course be made to follow.†

18. To a point on the sextic curve corresponds thus a line, and to any point on that line corresponds back again the point on the sextic. If then, a point P move along one of these lines the corresponding three axial pencils described by its polar planes have their axes meeting in the point on the sextic. The argument of paragraph 13 can be extended to show that if the axes of three projective pencils meet in a point there will be three sets of corresponding planes at most and one set at least that meet in the same straight line. This means that there are three positions for P on the line in question in which its three polar planes meet in a line. In other words, *Every line corresponding to a point on the sextic curve meets the sextic in one or three points.* From the proof we have established also that, *Through every point on the sextic curve will pass one or three lines of the sort in question.* It is of interest then to investigate the singly infinite system of lines which correspond to points on the Jacobian sextic of three quadrics.

19. To obtain the degree of the surface O of which these lines are the generators we observe first that a straight line must meet it in eight points. For the twisted cubic corresponding to the line must cut the Jacobian sextic eight times. Otherwise the twisted cubic would go back into a curve of degree higher than one. The curve of degree nine into which this twisted cubic must go is thus a degenerate curve made up of the original line and eight lines of the surface O. Again a plane goes into a cubic surface. This cubic surface contains every point of the sextic since the plane from which it is derived cuts every line of the surface O. The cubic surface goes back into a plane instead of into a surface of degree nine. This means that the surface O is of the eighth degree.

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* See Salmon's *Geometry of Three Dimensions*, p. 213.

† Reye, l. c., Chapter 26.

20. This surface O is thus a ruled surface of degree eight, every ruling of which meets the Jacobian in three points (or one) and such that three (or one) rulings go through every point on the sextic. The Jacobian is thus a triple line on the surface, three sheets of the surface passing through it. A plane section of the surface will thus give a curve of degree eight with six triple points.

21. This surface O and the Jacobian sextic of the three quadric surfaces will serve to explain the discrepancies in the degrees of reciprocal curves and surfaces. Thus a curve of degree n goes by the transfarmation into a curve of degree $3n$. But since the curve of degree n meets the surface O in $8n$ points, the curve of degree $3n$ meets the sextic in $8n$ points. When therefore, we reciprocate *this* curve of degree $3n$ part of the resulting curve is made up of $8n$ lines of the surface O, and the degree of the resultant curve is thus $9n-8n$ or n, which is the original curve. Similarly for surfaces. Every ruling of O meets a surface of degree n in n points, so that the reciprocal surface of degree $3n$ passes n times through the Jacobian curve. In reciprocating the surface back again we get a surface of degree $9n$ to be sure, but the surface O is n times a part of it. The reciprocal surface is thus of degree $9n-8n$ or n, the original surface.

PROOF OF THE FIRST FORMULA FOR EVALUATING 0/0.

By N. J. LENNES, Columbia University, New York.

It is well known that many proofs of theorems, while entirely logical and conclusive, afford little or no insight into the theorems proved and certainly furnish no obvious clue to the origin either of the theorem or the proof. On the other hand it is by no means infrequent that one has direct insight into the nature of a theorem or mathematical process and that such insight may be of great value both as an aid to the memory and in judging the true value and uses of the theorem or process while at the same time it may be found impossible, or at least impracticable, to formulate the elements of thought which constitute this insight into a logical proof.

It is believed to be a pedagogical principle of wide application that, other things being equal, the one of two modes of treatment of any topic is to be preferred which affords the more direct and obvious connection between intuitional insight on the one hand and rigorous logical proof on the other. For this reason the proofs given below seem to be worth publishing.

Theorem. *If* $f_1(x)$ *and* $f_2(x)$ *are continuous functions each possessing a derivative at* $x=a$, *and such that* $f_1(a)=0$, $f_2(a)=0$, *while* $f_2'(a)\neq 0$, *then*

$$\operatorname*{Lim.}_{x \doteq a} \frac{f_1(x)}{f_2(x)} = \frac{f_1'(a)}{f_2'(a)}.$$